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TECHNICAL NOTES

Transient laminar forced convection in ducts with suddenly applied uniform wall heat flux

W. S. KIM and M. N. ÖZIŞIK
Department of Mechanical and Aerospace Engineering, North Carolina State University,
Campus Box 7910, Raleigh, NC 27695-7910, U.S.A.

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INTRODUCTION

THERMAL transients in forced convection inside ducts have numerous applications in the design of control systems for heat exchangers. Only a limited amount of work is available [1–6] on this subject, and all such studies are concerned with the transients associated with variations in the wall surface temperature or the inlet temperature of the fluid.

In this work analytic solutions are developed for unsteady laminar forced convection inside circular tubes and parallel plate channels resulting from a step variation in the wall heat flux. The generalized integral transform technique [7] and the classical Laplace transformation are used to develop a simple lowest order solution as well as higher order solutions.

ANALYSIS

We consider transient forced convection of a laminar, incompressible, thermally developing and hydrodynamically developed Newtonian flow inside a circular tube or a parallel-plate channel subjected to a prescribed wall heat flux. Axial conduction and viscous dissipation are neglected, and the physical properties are considered constant. The mathematical formulation of the problem in dimensionless form is given by

$$\frac{\partial\theta(R,Z,\tau)}{\partial\tau}+W(R)\frac{\partial\theta(R,Z,\tau)}{\partial Z}=\frac{1}{R^p}\frac{\partial}{\partial R}\left[R^p\frac{\partial\theta(R,Z,\tau)}{\partial R}\right]$$

$$\text{in}\quad 0<R<R_w,\ Z>0,\ \tau>0 \qquad (1a)$$

$$\theta(R,Z,0)=0,\quad \theta(R,0,\tau)=0,\quad 0\leqslant R\leqslant R_w \qquad (1b,c)$$

$$\left.\frac{\partial\theta(R,Z,\tau)}{\partial R}\right|_{R=0}=0,\quad \left.\frac{\partial\theta(R,Z,\tau)}{\partial R}\right|_{R=R_w}=\frac{l_0}{l_N},$$

$$\tau>0,\ Z>0 \qquad (1d,e)$$

where, $W(R)$ is the dimensionless velocity profile

$$W(R)=\left[\frac{l_0}{l_N}\right]^2U(R),\quad U(R)=c\left[1-\left(\frac{l_0}{r_w}R\right)^2\right],$$

$$C=\frac{1+(2+p)}{2} \qquad (1f–h)$$

we split the problem into two parts as

$$\theta(R,Z,\tau)=\Phi(R,Z,\tau)+\theta_c(R,\tau) \qquad (2)$$

where $\theta_c(R,\tau)$ is the solution of the following transient conduction problem

$$\frac{\partial\theta_c(R,\tau)}{\partial\tau}=\frac{1}{R^p}\frac{\partial}{\partial R}\left[R^p\frac{\partial\theta_c(R,\tau)}{\partial R}\right],\quad \tau>0,\ 0<R<R_w \qquad (3a)$$

$$\theta_c(R,\tau)=0\quad\text{at}\quad\tau=0\quad\text{in}\quad 0\leqslant R\leqslant R_w \qquad (3b)$$

$$\left.\frac{\partial\theta_c(R,\tau)}{\partial R}\right|_{R=0}=0,\quad \left.\frac{\partial\theta_c(R,\tau)}{\partial R}\right|_{R=R_w}=\frac{l_0}{l_N},\quad \tau>0 \qquad (3c,d)$$

and $\Phi(R,Z,\tau)$ satisfies the following system

$$\frac{\partial\Phi(R,Z,\tau)}{\partial\tau}+W(R)\frac{\partial\Phi(R,Z,\tau)}{\partial Z}=\frac{1}{R^p}\frac{\partial}{\partial R}\left[R^p\frac{\partial\Phi(R,Z,\tau)}{\partial R}\right],$$

$$Z>0,\ \tau>0,\ 0<R<R_w \qquad (4a)$$

$$\Phi(R,Z,\tau)=0\quad\text{at}\quad\tau=0\quad\text{in}\quad Z\geqslant 0,\ 0\leqslant R\leqslant R_w \qquad (4b)$$

$$\Phi(R,Z,\tau)=-\theta_c(R,\tau)\quad\text{at}\quad Z=0$$

$$\text{in}\quad \tau>0,\ 0\leqslant R\leqslant R_w \qquad (4c)$$

$$\left.\frac{\partial\Phi(R,Z,\tau)}{\partial R}\right|_{R=0}=0,\quad \left.\frac{\partial\Phi(R,Z,\tau)}{\partial R}\right|_{R=R_w}=0,$$

$$\tau>0,\ Z>0. \qquad (4d,e)$$

The solution of the pure conduction problem (3) is readily obtained as

$$\theta_c(R,\tau)=B\tau+\sum_{m=1}^{\infty}\frac{1}{N_m^*(\gamma_m)}F_m^*X_m(\gamma_m,R)(1-e^{-\gamma_m^2\tau}) \qquad (5a)$$

where

$$B=\frac{(p+1)(l_0/l_N)}{R_w},\quad F_m^*=R_w^p(l_0/l_N)\frac{X_m(\gamma_m,R_w)}{\gamma_m^2},$$

$$N_m^*=\int_0^{R_w}R^p[X_m(\gamma_m,R)]^2\,dR \qquad (5b–d)$$

and $X_m(\gamma_m,R)$ are the eigenfunctions and γ_m the eigenvalues of the eigenvalue problem appropriate for system (3).

To solve problem (4), we successively take the Laplace transform with respect to the τ variable and integral transform with respect to the R variable to obtain an infinite system of ordinary differential equations in the Z variable for the double transform $\bar{\bar{\Phi}}_i(Z)$, $i=1,2,3,\ldots$ of the function $\Phi(R,Z,\tau)$.

A lowest order solution is obtained by retaining only one term in the summation in the system of ordinary differential equation for $\bar{\bar{\Phi}}_i(Z)$. The solution of this equation, after successive inversion with respect to the R and τ variables, gives the lowest order solution $\Phi_1(R,Z,\tau)$ for system (4) as

$$\Phi_1(R,Z,\tau)=-\sum_{i=0}^{\infty}\frac{\psi_i(\beta_i,R)}{N_i}e^{-\beta_i^2Z}\,U(\tau-A_{ii}Z)$$

$$\times\left[B(\tau-A_{ii}Z)F_i+\sum_{m=1}^{\infty}(1-e^{-\gamma_m^2(\tau-A_{ii}Z)})F_{mi}\right] \qquad (6a)$$

NOMENCLATURE

D_h	hydraulic diameter, $4b$ for parallel-plate duct, $2b$ for circular tube
l_0, l_N	reference lengths to nondimensionalize r- and z-coordinates, respectively ($l_0 = b$, $l_N = D_h$)
$Nu(Z,\tau)$	local Nusselt number
p	0 for parallel-plate channel, 1 for circular tube
q_w	prescribed wall heat flux
r_w, R_w	radius of circular duct or half the spacing between parallel plates, dimensional and dimensionless, respectively
R	dimensionless radial coordinate, r/l_0
s	Laplace transform variable
T_i	initial and inlet fluid temperature
$U(R)$	normalized velocity profile, $w(r)/w_{av}$
w_{av}	average velocity
Z	dimensionless axial coordinate, $\alpha z/w_{av}l_N^2$.

Greek symbols

α	thermal diffusivity
$\theta(R,Z,\tau)$	dimensionless temperature, $(T(r,z,t)-T_i)/(q_w l_N/k)$
τ	dimensionless time, $\alpha t/l_0^2$.

where

$$A_{ii} = \frac{1}{N_i}\int_0^{R_w} R^p[\psi_i(\beta_i,R)]^2\,dR,$$

$$N_i = \int_0^{R_w} R^p W(R)[\psi_i(\beta_i,R)]^2\,dR \qquad (6b,c)$$

$$F_i = \int_0^{R_w} R^p W(R)\psi_i(\beta_i,R)\,dR = -\frac{1}{\beta_i^2}R_w^p\psi_i'(\beta_i,R_w),$$
$$\beta_i \neq 0,\ i = 1,2,\ldots \qquad (6d)$$

$$F_0 = \int_0^{R_w} R^p W(R)\,dR \qquad (6e)$$

$$F_{mi} = \frac{F_m^*}{N_m^*}\int_0^{R_w} R^p W(R)X_m(\gamma_m,R)\psi_i(\beta_i,R)\,dR$$
$$\text{with}\quad \psi_0(\beta_0,R) = 1 \qquad (6f)$$

$$U(\tau - A_{ii}Z) = \begin{cases} 1 & \text{for} \quad \tau > A_{ii}Z \\ 0 & \text{for} \quad \tau < A_{ii}Z \end{cases} \qquad (6g)$$

and $\psi_i(\beta_i,R)$ are the eigenfunctions and β_i the eigenvalues of the eigenvalue problem in the R variable appropriate to the Laplace transform in the τ variable of system (4).

A higher order solution can be obtained by retaining more terms in the summation; but, it makes the system of equations too involved to be useful for practical purposes. However, a straight-forward higher order solution can be obtained by following a procedure similar to that described in ref. [6]. The higher order solution for $\Phi(R, Z, \tau)$ can be written as

$$\Phi_h(R,Z,\tau) = \Phi_l(R,Z,\tau) - \Phi_c(R,Z,\tau) \qquad (7)$$

where $\Phi_l(R, Z, \tau)$ is already given by equation (6a) and $\Phi_c(R, Z, \tau)$ is a correction term given by

$$\Phi_c(R,Z,\tau) = \sum_{j=1}^{\infty}\frac{\psi_0(\beta_0,R)}{N_0}A_{0j}S_j(Z,\tau) \qquad (8a)$$

where

$$S_j(Z,\tau) = -\frac{BF_j}{A_j^*\lambda_j^*}e^{-\beta_j^2 Z}\,U(\tau - A_{jj}Z)[1 - e^{-\lambda_j^*(\tau - A_{jj}Z)}]$$
$$+\frac{BF_j}{A_j^*\lambda_j^*}e^{-\beta_0^2 Z}\,U(\tau - A_{00}Z)[1 - e^{-\lambda_j^*(\tau - A_{00}Z)}]$$
$$+\frac{e^{-\beta_j^2 Z}}{A_j^*}U(\tau - A_{jj}Z)\sum_{m=1}^{\infty}\frac{F_{mj}}{\lambda_j^* - \gamma_m^2}$$
$$\times[\lambda_j^*\,e^{-\lambda_j^*(\tau - A_{jj}Z)} - \gamma_m^2\,e^{-\gamma_m^2(\tau - A_{jj}Z)}]$$
$$-\frac{e^{-\beta_0^2 Z}}{A_j^*}U(\tau - A_{00}Z)\sum_{m=1}^{\infty}\frac{F_{mj}}{\lambda_j^* - \gamma_m^2}$$
$$\times[\lambda_j^*\,e^{-\lambda_j^*(\tau - A_{00}Z)} - \gamma_m^2\,e^{-\gamma_m^2(\tau - A_{00}Z)}]$$
$$-\frac{e^{-\beta_j^2 Z}}{A_j^*}U(\tau - A_{jj}Z)\,e^{-\lambda_j^*(\tau - A_{jj}Z)}\sum_{m=1}^{\infty}F_{mj}$$
$$+\frac{e^{-\beta_0^2 Z}}{A_j^*}U(\tau - A_{00}Z)\,e^{-\lambda_j^*(\tau - A_{00}Z)}\sum_{m=1}^{\infty}F_{mj} \qquad (8b)$$

and

$$A_j^* = A_{00} - A_{jj}, \quad \lambda_j^* = \frac{\beta_0^2 - \beta_j^2}{A_j^*}, \quad \beta_0 = 0. \qquad (8c)$$

Then, the complete lowest order solution $\theta_l(R, Z, \tau)$ becomes

$$\theta_l(R,Z,\tau) = -\sum_{i=0}^{\infty}\frac{\psi_i(\beta_i,R)}{N_i}e^{-\beta_i^2 Z}\,U(\tau - A_{ii}Z)$$
$$\times\left[B(\tau - A_{ii}Z)F_i + \sum_{m=1}^{\infty}\cdot(1 - e^{-\gamma_m^2(\tau - A_{ii}Z)})F_{mi}\right]$$
$$+B\tau + \sum_{m=1}^{\infty}\frac{F_m^*}{N_m^*}X_m(\gamma_m,R)(1 - e^{-\gamma_m^2\tau}) \qquad (9)$$

and, the higher order solution $\theta_h(R, Z, \tau)$ is determined by adding the correction term to $\theta_l(R, Z, \tau)$ as

$$\theta_h(R,Z,\tau) = \theta_l(R,Z,\tau) - \sum_{j=1}^{\infty}\frac{\psi_0(\beta_0,R)}{N_0}A_{0j}S_j(Z,\tau). \qquad (10)$$

Using the lowest order analytical solution, the dimensionless average flow temperature and local Nusselt number are determined, respectively, as

$$\theta_{av,l}(Z,\tau) = \left(\frac{l_N}{l_0}\right)^2\frac{p+1}{R_w^{p+1}}\left\{-\sum_{i=0}^{\infty}\frac{F_i}{N_i}e^{-\beta_i^2 Z}\,U(\tau - A_{ii}Z)\right.$$
$$\times\left[B(\tau - A_{ii}Z)F_i + \sum_{m=1}^{\infty}(1 - e^{-\gamma_m^2(\tau - A_{ii}Z)})F_{mi}\right]$$
$$\left.+B\tau\int_0^{R_w}R^p W(R)\,dR + \sum_{m=1}^{\infty}\frac{F_m^*}{N_m^*}(1 - e^{-\gamma_m^2\tau})J_m\right\}$$

where

$$J_m = \int_0^{R_w}R^p W(R)X_m(\gamma_m,R)\,dR \qquad (11)$$

$$Nu_l(Z,\tau) = \frac{1}{\theta_l(R_w,Z,\tau) - \theta_{av,l}(Z,\tau)}. \qquad (12)$$

Similarly, using the higher order solution the corresponding quantities are

$$\theta_{av,h}(Z,\tau) = \theta_{av,l}(Z,\tau) - \left(\frac{l_N}{l_0}\right)^2\frac{p+1}{R_w^{p+1}}$$
$$\times\sum_{j=1}^{\infty}\frac{A_{0j}}{N_0}F_0\cdot S_j(Z,\tau) \qquad (13)$$

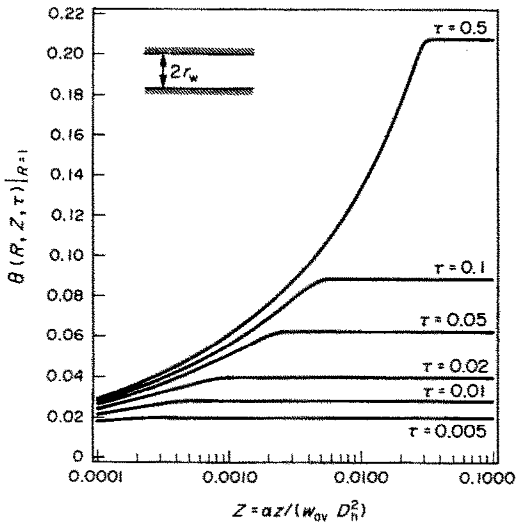

FIG. 1. Dimensionless wall temperature for a parallel-plate channel at different dimensionless times.

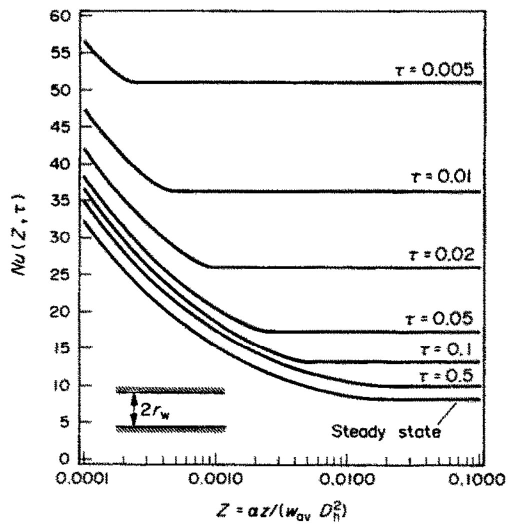

FIG. 3. Local Nusselt number for a parallel-plate channel at different dimensionless times.

$$Nu_h(Z, \tau) = \frac{1}{\theta_h(R_w, Z, \tau) - \theta_{av,h}(Z, \tau)}. \qquad (14)$$

The sign-count method [8–10] is used to solve the eigenvalue problems.

RESULTS AND DISCUSSION

We now examine the thermal response of laminar flow inside a parallel-plate channel and a circular tube for a suddenly applied uniform wall heat flux. For computational purposes, the reference lengths are taken as $l_0 = r_w$ and $l_N = D_h$, where D_h is the hydraulic diameter of the conduit under consideration. Then the dimensionless velocity profile becomes

$$W(R) = C^*[1 - R^2],$$

$$C^* = \left(\frac{l_0}{l_N}\right)^2 \frac{1 + (2+p)}{2} = \begin{cases} \frac{3}{32} \text{parallel plate} \\ \frac{1}{2} \text{circular tube.} \end{cases} \qquad (15)$$

Once the eigenvalues, eigenfunctions and the normalization integrals are available, the temperature distribution in the flow and the Nusselt number are readily computed from the expressions given previously.

Here we consider results obtained from the lowest order solution, since the explicit form is useful for evaluating the Nusselt number readily, and the fact that the lowest order solution, for most practical purposes, sufficiently accurate as pointed out by Cotta and Özişik [6] for a related problem.

Figures 1 and 2 show the dimensionless wall temperature for a parallel-plate channel and a circular tube, respectively, plotted as a function of the dimensionless axial coordinate in the range $10^{-4} \leqslant Z \leqslant 10^{-1}$ for several different values of the dimensionless time τ. Starting from the inlet, the wall temperature increases monotonically with both time and position along the conduit until the location characterizing the beginning of the conduction region is reached. In the conduction region the wall temperature remains invariant with the axial distance but increases continuously with time. At a given time and position along the duct, the wall temperature for a circular tube is higher than that for a parallel-plate channel.

Figures 3 and 4 show the local Nusselt numbers for a

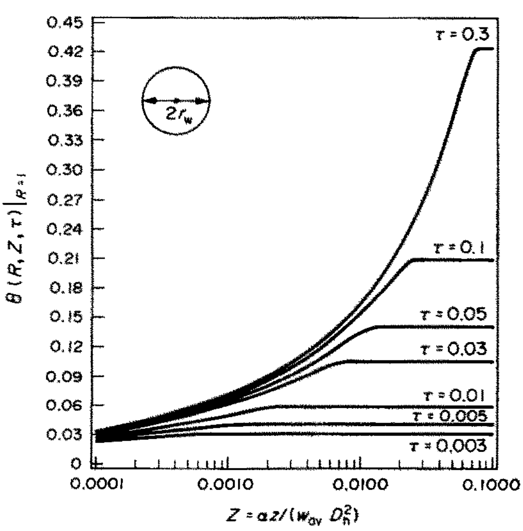

FIG. 2. Dimensionless wall temperature for a circular tube at different dimensionless times.

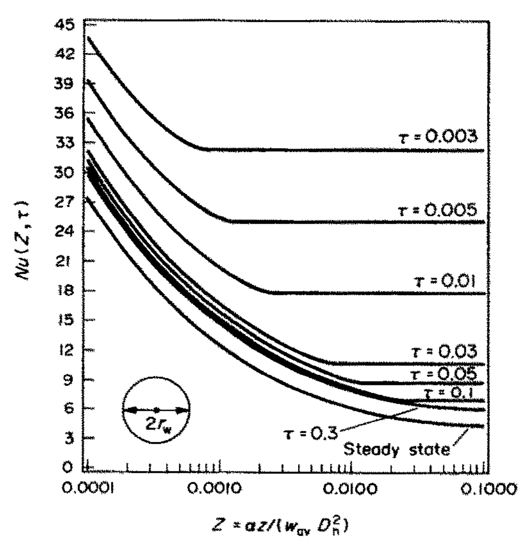

FIG. 4. Local Nusselt number for a circular tube at different dimensionless times.

parallel-plate channel and a circular tube, respectively, plotted against the axial distance in the range $10^{-4} \leqslant Z \leqslant 10^{-1}$ for several different values of the dimensionless time τ. Starting from the inlet region, the local Nusselt numbers decrease continuously with both increasing time and axial location along the conduit until the conduction region is reached. In the conduction region, the Nusselt number remains invariant with the position but decreases with increasing time. Eventually, with increasing time, the local Nusselt numbers for both regions assume the well-known steady-state value.

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Natural convection heat transfer in enclosures with an off-center partition

TATSUO NISHIMURA, MITSUHIRO SHIRAISHI and YUJI KAWAMURA
Department of Chemical Engineering, Hiroshima University, Higashi-Hiroshima 724, Japan

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1. INTRODUCTION

NATURAL convection through externally heated and cooled enclosures is of interest in solar collector applications, in the estimation of heat losses from double-pane windows, and in the calculation of heat losses from rooms. Numerous experimental and numerical computational studies have been reported explaining the heat transfer mechanism and presenting correlations for heat transfer rates for such systems. Excellent reviews [1, 2] are available and there is no need to repeat them here.

The problem of primary interest in the literature [1, 2] is that of an enclosure with no partitions. However, in practical cases, a vertical partition is inserted into the enclosure to reduce heat losses by natural convection and thermal radiation. Reported studies of natural convection in a partitioned enclosure are limited. Duxbury [3] reported experiments with air-filled enclosures containing a central partition as shown in Fig. 1. Nakamura *et al.* [4] performed computational and experimental studies including the effect of thermal radiation for the same configuration as that of Duxbury. The present authors [5] proposed a boundary layer solution for this system and confirmed its validity by experiments. Tong and Gerner [6] reported the effect of partition position on the heat transfer rate by numerical computation and concluded that a central partition corresponding to $W'/W = 0.5$ produces the greatest reduction in heat transfer.

This study is an extension of the previous study [5]. We examine the limitations of the boundary layer approximation for various positions of the partition. We show that even if the partition deviates from the center of the enclosure, the heat transfer rate is identical with that for the partition in the central position. This does not appear to have been studied previously.

2. LIMITATIONS OF THE BOUNDARY LAYER APPROXIMATION

We [5] previously indicated that a thermal boundary layer with a constant thickness is developed along the partition at high Rayleigh numbers for the enclosure with a central

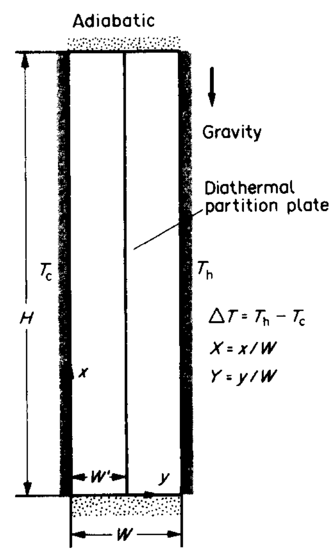

FIG. 1. Schematic diagram of an enclosure divided by a vertical partition.